Journal of Approximation Theory **111**, 267–297 (2001)
doi:10.1006/jath.2001.3577, available online at http://www.idealibrary.com on IDEAL®

Stable Local Bases for Multivariate Spline Spaces

Oleg Davydov

Mathematical Institute, Justus Liebig University, D-35392 Giessen, Germany
E-mail: oleg.davydov@math.uni-giessen.de

Communicated by Carl de Boor

Received March 14, 2000; accepted in revised form February 20, 2001;
published online June 18, 2001

We present an algorithm for constructing stable local bases for the spaces $\mathcal{S}^r_d(\Delta)$ of multivariate polynomial splines of smoothness $r \geqslant 1$ and degree $d \geqslant r2^n + 1$ on an arbitrary triangulation Δ of a bounded polyhedral domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$.

1. INTRODUCTION

Let Δ be a *triangulation* of a bounded polyhedral domain $\Omega \subset \mathbb{R}^n$, i.e., Δ is a finite set of non-degenerate n-simplices such that

(1) $\Omega = \bigcup_{T \in \Delta} T$;

(2) the interiors of the simplices in Δ are pairwise disjoint; and

(3) each facet of a simplex $T \in \Delta$ either lies on the boundary of Ω or is a common face of exactly two simplices in Δ.

Given $1 \leqslant r \leqslant d$, we consider the *spline space*

$$\mathcal{S}^r_d(\Delta) := \{ s \in C^r(\Omega) : s|_T \in \Pi^n_d \text{ for all } n\text{-simplices } T \in \Delta \},$$

where Π^n_d is the linear space of all n-variate polynomials of total degree at most d. It is well-known that $\dim \Pi^n_d = \binom{n+d}{n}$.

The application of splines in numerical computations requires efficient algorithms for constructing locally supported bases for the space $\mathcal{S}^r_d(\Delta)$ or its subspaces (such as finite element spaces). Moreover, if a *local* basis $\{s_1, ..., s_m\}$ for $\mathcal{S}^r_d(\Delta)$ is in addition *stable*, i.e., for all $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$,

$$K_1 \|\alpha\|_{\ell_p} \leqslant \Big\| \sum_{k=1}^m \alpha_k s_k \Big\|_{L_p(\Omega)} \leqslant K_2 \|\alpha\|_{\ell_p},$$

0021-9045/01 $35.00

then a *nested* sequence of spaces

$$\mathcal{S}^r_d(\Delta_1) \subset \mathcal{S}^r_d(\Delta_2) \subset \cdots \subset \mathcal{S}^r_d(\Delta_q) \subset \cdots, \tag{1.1}$$

may be used for designing multilevel methods of approximation on a bounded domain $\Omega \subset \mathbb{R}^n$, see e.g. [27] and references therein. In particular, the sequence (1.1) constitutes a *multiresolution analysis* on Ω if the maximal diameter of the triangles in Δ_q tends to zero as $q \to \infty$, and if the constants $0 < K_1, K_2 < \infty$ are independent of q. Note that the bases for the *full space* $\mathcal{S}^r_d(\Delta)$ are particularly interesting since $\mathcal{S}^r_d(\Delta_q) \subset \mathcal{S}^r_d(\Delta_{q+1})$ if Δ_{q+1} is a *refinement* of Δ_q. (This is not the case for the finite element subspaces of $\mathcal{S}^r_d(\Delta)$ when $r \geqslant 1$; see [14, 25, 27].)

The famous *B*-splines constitute a stable locally supported basis for the space $\mathcal{S}^r_d(\Delta)$ in the one-dimensional case $n = 1$ for all $d \geqslant r + 1$. Moreover, the dual basis is also local and therefore provides a quasi-interpolant possessing optimal approximation order. There are well known constructions of local bases for $\mathcal{S}^r_d(\Delta)$ in the bivariate case $n = 2$ for all $d \geqslant 3r + 2$, see [1, 21, 22, 26]. Stable local bases were constructed in [7, 23] for some superspline subspaces, and in [17, 19] for the full bivariate spline spaces $\mathcal{S}^r_d(\Delta)$, $d \geqslant 3r + 2$. In the trivariate case $n = 3$ local bases are known for all $d \geqslant 8r + 1$ [2]. It was conjected in [2] that in general locally supported bases for $\mathcal{S}^r_d(\Delta)$ exist if $d \geqslant r(2^n - 1) + n$.

The main objective of this paper is to construct stable locally supported bases for $\mathcal{S}^r_d(\Delta)$ and its superspline subspaces for all $n \geqslant 2$ and $r \geqslant 1$ provided $d \geqslant r2^n + 1$.

We make use of the *nodal approach* originated in the finite element method, see e.g. [12], and extended to the problems of spline spaces on general triangulations in [26] and more recently in [8–11, 15, 16, 17]. We show that in the multivariate case the *nodal smoothness conditions* can be better localized than usual Bernstein–Bézier smoothness conditions [5, 20]. The key point for our analysis is that certain matrices associated with the smoothness conditions have a block diagonal structure, which in the same time makes it possible to handle them efficiently in numerical computations, see Sections 5 and 6. In particular, the dimension of any given spline space $\mathcal{S}^r_d(\Delta)$, $d \geqslant r2^n + 1$, can be efficiently computed by a formula obtained in Section 5.

The paper is organized as follows. In Section 2 we give some definitions and preliminary lemmas. The nodal functionals that we use are described in Section 3. Section 4 is devoted to a detailed analysis of nodal smoothness conditions. In Section 5 we construct local bases for $\mathcal{S}^r_d(\Delta)$, $d \geqslant r2^n + 1$. In Section 6 we show how to achieve stability of these bases. Finally, in Section 7 we extend the results to the superspline subspaces of $\mathcal{S}^r_d(\Delta)$.

2. PRELIMINARIES

2.1. *Bases and Minimal Determining Sets*

It is obvious that the linear space $\mathscr{S}^r_d(\Delta)$ has finite dimension. In this subsection we consider an abstract finite-dimensional linear space $\mathscr{S}$, although in all our applications we have $\mathscr{S} \subset \mathscr{S}^r_d(\Delta)$.

Let $\mathscr{S}^*$ denote, as usual, the dual space of linear functionals on $\mathscr{S}$. Given a basis $\{s_j\}_{i \in J}$ for $\mathscr{S}$, its *dual basis* is a basis $\{\lambda_j\}_{j \in J}$ for $\mathscr{S}^*$ such that

$$\lambda_i s_j = \delta_{i,j}, \qquad \text{all} \quad i, j \in J. \tag{2.1}$$

It is easy to see that the dual basis $\{\lambda_j\}_{j \in J}$ is uniquely determined by $\{s_j\}_{j \in J}$, and vice versa, a basis $\{\lambda_j\}_{j \in J}$ for $\mathscr{S}^*$ uniquely determines a basis $\{s_j\}_{j \in J}$ for $\mathscr{S}$ satisfying (2.1).

In order to construct a basis $\{s_j\}_{j \in J}$ for a spline space $\mathscr{S}$ it is often useful to find first a basis $\{\lambda_j\}_{j \in J}$ for $\mathscr{S}^*$ and then determine $\{s_j\}_{j \in J}$ from the duality condition (2.1). Usually, the required basis for $\mathscr{S}^*$ can be selected by an algorithm from a larger set $\Lambda \subset \mathscr{S}^*$ that spans $\mathscr{S}^*$. A common example of such a set Λ is the set of linear functionals picking off a coefficient of the Bernstein–Bézier representation of splines $s \in \mathscr{S}$, see e.g. [2]. Keeping in mind the tradition upheld in the literature on bivariate and multivariate splines, we will use the following terminology.

DEFINITION 2.1. Any finite spanning set for $\mathscr{S}^*$ is called a *determining set* for $\mathscr{S}$. Any basis for $\mathscr{S}^*$ is called a *minimal determining set* for $\mathscr{S}$.

A standard argument in linear algebra shows that a set $\Lambda \subset \mathscr{S}^*$ is a determining set for $\mathscr{S}$ if and only if $\lambda s = 0$ for all $\lambda \in \Lambda$ implies $s = 0$ whenever $s \in \mathscr{S}$. Moreover, a determining set Λ is a minimal determining set for $\mathscr{S}$ if and only if no proper subset of Λ is a determining set. Since every linear functional on $\mathscr{S}$ is well-defined on any subspace $\tilde{\mathscr{S}}$ of $\mathscr{S}$, it is easy to see that a determining set for $\mathscr{S}$ is also a determining set for $\tilde{\mathscr{S}}$.

Suppose Λ is a determining set for $\mathscr{S}$. If Λ is not a minimal determining set for $\mathscr{S}$, then Λ is linearly dependent. It is particularly useful to know a complete system of linear relations for Λ.

DEFINITION 2.2. Let $\Lambda = \{\lambda_j\}_{j \in J} \subset \mathscr{S}^*$ be a determining set for $\mathscr{S}$. Suppose that the functionals λ_j satisfy linear conditions

$$\sum_{j \in J} c_{i,j} \lambda_j = 0, \qquad i \in I, \tag{2.2}$$

where $c_{i,j}$ are some real coefficients. We say that (2.2) is a *complete system of linear relations for Λ over $\mathscr{S}$* if for any $a=(a_j)_{j\in J}$, with $a_j\in\mathbb{R}$, $j\in J$, such that

$$\sum_{j\in J} c_{i,j}a_j=0, \qquad i\in I, \tag{2.3}$$

there exists an element $s\in\mathscr{S}$ such that $\lambda_j s=a_j$ for all $j\in J$.

Note that the element $s\in\mathscr{S}$ as above is necessarily *unique*. Indeed, if there are $s_1, s_2\in\mathscr{S}$ such that $\lambda_j s_1=\lambda_j s_2=a_j$ for all $j\in J$, then $\lambda_j(s_1-s_2)=0$, $j\in J$, which implies $s_1=s_2$ since Λ is a determining set for $\mathscr{S}$.

Let $C:=(c_{i,j})_{i\in I, j\in J}$. Then (2.3) means that the vector a lies in the null space $N(C):=\{a\colon Ca^T=0\}$ of the matrix C. Thus, there is a 1–1 correspondence between elements $s\in\mathscr{S}$ and vectors $a\in N(C)$, where $a=(a_j)_{j\in J}$, $a_j=\lambda_j s$. In particular, the dimension of $\mathscr{S}$ can be computed as follows.

LEMMA 2.3. *We have*

$$\dim\mathscr{S}=\dim N(C)=\#\Lambda-\operatorname{rank} C. \tag{2.4}$$

Moreover, given a determining set Λ for $\mathscr{S}$ and a complete system of linear relations for Λ over $\mathscr{S}$ with matrix C, it is straightforward to construct a basis for $\mathscr{S}$; see also [6].

ALGORITHM 2.4. Suppose $\Lambda=\{\lambda_j\}_{j\in J}\subset\mathscr{S}^*$ is a determining set for $\mathscr{S}$, and (2.2) is a complete system of linear relations for Λ over $\mathscr{S}$. Let $a^{[k]}=(a_j^{[k]})_{j\in J}$, $k=1, ..., m$, form a basis for the null space $N(C)$ of C. For each $k=1, ..., m$, construct the unique element $\tilde{s}_k\in\mathscr{S}$ satisfying $\lambda_j\tilde{s}_k=a_j^{[k]}$ for all $j\in J$. Then $\{\tilde{s}_1, ..., \tilde{s}_m\}$ is a basis for $\mathscr{S}$.

It is not difficult to determine corresponding minimal determining set, i.e., the basis $\{\tilde{\lambda}_1, ..., \tilde{\lambda}_m\}$ for $\mathscr{S}^*$ dual to $\{\tilde{s}_1, ..., \tilde{s}_m\}$. Let

$$A:=[a_j^{[k]}]_{j\in J, k=1, ..., m}.$$

Since the columns $a^{[k]}$ of this matrix are linearly independent, A has full column rank. Hence, there exists a left inverse of A, i.e., a matrix

$$B=[b_{k,j}]_{k=1, ..., m, j\in J}$$

satisfying $BA=I_m$, where I_m is the $m\times m$ identity matrix. Note that B is not unique in general.

LEMMA 2.5. *The dual basis* $\{\tilde{\lambda}_1, ..., \tilde{\lambda}_m\}$ *can be computed by*

$$\tilde{\lambda}_k = \sum_{j \in J} b_{k,j} \lambda_j, \qquad k = 1, ..., m.$$

Proof. It is straightforward to check that the duality condition (2.1) is satisfied. ∎

2.2. *Geometry of a Triangulation in* $\mathbb{R}^n$

Recall that an ℓ-*simplex* τ $(0 \leqslant \ell \leqslant n)$ is the convex hull $\langle v_0, ..., v_\ell \rangle$ of $\ell + 1$ points $v_0, ..., v_\ell \in \mathbb{R}^n$ called *vertices* of τ. The simplex τ is *non-degenerate* if its ℓ-dimensional volume is non-zero and *degenerate* otherwise. The *dimension* of a non-degenerate ℓ-simplex is ℓ. By the *interior* of an ℓ-simplex we mean its ℓ-dimensional interior. The convex hull of a subset of $\{v_0, ..., v_\ell\}$ containing $m + 1 \leqslant \ell + 1$ elements is an *m-face* of τ. Thus, an m-face is itself an m-simplex. An $(\ell - 1)$-face of τ is also called a *facet* of τ, and any 1-face of τ is also called an *edge* of τ. Note that the only ℓ-face of τ is τ itself, and the vertices of τ are its 0-faces. (We identify a vertex v and its convex hull $\{v\}$.)

Denote by $\mathcal{T}_\ell$ the set of all ℓ-faces of the simplices in Δ $(\ell = 0, ..., n - 1)$ and set

$$\mathcal{T} := \bigcup_{\ell = 0}^{n} \mathcal{T}_\ell,$$

where $\mathcal{T}_n := \Delta$. We will also use notation $\mathcal{V} := \mathcal{T}_0$, $\mathcal{E} := \mathcal{T}_1$ and $\mathcal{F} := \mathcal{T}_{n-1}$ for the sets of all vertices, edges and facets of Δ, respectively. The *star* of a simplex $\tau \in \mathcal{T}$, denoted by star (τ), is the union of all n-simplices $T \in \Delta$ containing τ, i.e.,

$$\operatorname{star}(\tau) = \bigcup_{\substack{T \in \Delta \\ \tau \subset T}} T.$$

In particular, $\operatorname{star}(T) = T$ for each $T \in \Delta$.

Furthermore, given $\tau \in \mathcal{T}_\ell$, $\ell \leqslant n - 1$, we denote by (τ) the linear manifold in $\mathbb{R}^n$ parallel to the affine span aff(τ) of τ and by $(\tau)^\perp$ the orthogonal complement of (τ) in $\mathbb{R}^n$. Note that $\dim(\tau)^\perp = n - \ell$. In particular, $(v)^\perp = \mathbb{R}^n$ for all $v \in \mathcal{V}$.

Let $\tau = \langle v_0, ..., v_\ell \rangle \in \mathcal{T}_\ell$, $\ell \leqslant n - 1$, and let $w \in \mathcal{V}$ be such that $\tau' = \langle \tau, w \rangle := \langle v_0, ..., v_\ell, w \rangle$ is in $\mathcal{T}_{\ell+1}$. Since $\dim(\tau)^\perp = n - \ell$ and $\dim(\tau') = \ell + 1$, the linear manifold $(\tau)^\perp \cap (\tau')$ has dimension 1. Moreover, since

$\mathrm{aff}(\tau)$ has codimension 1 as an affine subspace of $\mathrm{aff}(\tau')$, it defines two half-spaces of $\mathrm{aff}(\tau')$, and there is a unique unit vector in $(\tau)^{\perp} \cap (\tau')$ pointing into the half-space of $\mathrm{aff}(\tau')$ containing w. We denote this unit vector by

$$\sigma_{\tau, w}.$$

If v is a vertex in $\mathscr{V}$, then $\sigma_{v, w}$ is obviously the unit vector in the direction of the edge $\langle v, w \rangle$. If $w_1, ..., w_m \in \mathscr{V}$ and $\tilde{\tau} = \langle \tau, w_1, ..., w_m \rangle$ is in $\mathscr{T}_{\ell+m}$, $\ell + m \leqslant n$, then we set

$$\sigma(\tau, \tilde{\tau}) := (\sigma_{\tau, w_1}, ..., \sigma_{\tau, w_m}).$$

2.3. *Nodal Functionals*

Given $\sigma = (\sigma_1, ..., \sigma_m)$ a linearly independent sequence of *unit* vectors in $\mathbb{R}^n$, and $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{Z}^m_+$, let D^{α}_{σ} denote the partial derivative

$$D^{\alpha}_{\sigma} := D^{\alpha_1}_{\sigma_1} \cdots D^{\alpha_m}_{\sigma_m},$$

where D_{σ_i} is the derivative in the direction σ_i,

$$D_{\sigma_i} f(x) := \lim_{t \to +0} t^{-1} \{ f(x + \sigma_i t) - f(x) \},$$

for a differentiable f. By a *nodal functional* we mean any linear functional on $\mathscr{S}^r_d(\Delta)$ of the form $\eta = \delta_x D^{\alpha}_{\sigma}$, where x is a point in Ω, and δ_x is the point-evaluation functional,

$$\delta_x f := f(x).$$

We denote by

$$q(\eta) = |\alpha| := \sum_{i=1}^{m} \alpha_i \leqslant r \tag{2.5}$$

the *order* of η. Given $s \in \mathscr{S}^r_d(\Delta)$, the partial derivative $D^{\alpha}_{\sigma} s$ is continuous everywhere in Ω if $|\alpha| \leqslant r$, and piecewise continuous if $|\alpha| > r$. In this last case we have to choose an n-simplex $T \in \Delta$, with $x \in T$, and apply our functional to $s|_T$. The following situation is of special interest since, for it, a *natural* choice for T exists. Assume that for some $\tau \in \mathscr{T}$ we have $x \in \tau$ and $x + \varepsilon \sigma_i \in \tau$, $i = 1, ..., m$, if $\varepsilon > 0$ is small enough. Then $\delta_x D^{\alpha}_{\sigma} s|_T$ is the same for all $T \in \Delta$ such that $\tau \subset T$. We will choose T in this way whenever the above situation occurs.

We will often use the following simple lemma.

LEMMA 2.6. *Let L be a linear manifold in $\mathbb{R}^n$, $\dim L = m \leqslant n$, and let $\sigma = (\sigma_1, ..., \sigma_m)$ be a basis of L, where $\sigma_1, ..., \sigma_m \in L$ are unit vectors. Suppose that all components of $\tilde{\sigma} = (\tilde{\sigma}_1, ..., \tilde{\sigma}_m)$ are also some unit vectors in L. Then for any $\alpha \in \mathbb{Z}^m$ there exist real coefficients c_β such that*

$$D^\alpha_{\tilde{\sigma}} = \sum_{\substack{\beta \in \mathbb{Z}^m \\ |\beta| = |\alpha|}} c_\beta D^\beta_\sigma.$$

Proof. Since σ is a basis for L, there are real coefficients a_{ij} such that

$$\tilde{\sigma}_i = \sum_{j=1}^m a_{ij} \sigma_j \qquad i = 1, ..., m.$$

Therefore,

$$D_{\tilde{\sigma}_i} = \sum_{j=1}^m a_{ij} D_{\sigma_j} \qquad i = 1, ..., m,$$

and

$$D^\alpha_{\tilde{\sigma}} = \left(\sum_{j=1}^m a_{1j} D_{\sigma_j} \right)^{\alpha_1} \cdots \left(\sum_{j=1}^m a_{mj} D_{\sigma_j} \right)^{\alpha_m},$$

where $\alpha = (\alpha_1, ..., \alpha_m)$. ∎

2.4. *Polynomial Unisolvent Sets*

Let τ be a non-degenerate ℓ-simplex in $\mathbb{R}^n$. We set

$$\Pi^\ell_m(\tau) := \{ p|_\tau : p \in \Pi^n_m \}, \qquad m = -1, 0, 1, 2, ...,$$

where Π^n_m is the space of all n-variate polynomials of total degree at most m, $m = 0, 1, 2, ...,$ and $\Pi^n_{-1} := \{0\}$. By a change of variables, the elements of $\Pi^\ell_m(\tau)$ may be considered as ℓ-variate polynomials of total degree at most m defined on τ. In particular, $\dim \Pi^\ell_m(\tau) = \dim \Pi^\ell_m = \binom{\ell+m}{m}$, $m = 0, 1, 2, ...,$ $\dim \Pi^\ell_{-1}(\tau) = 0$. A finite set $\Xi \subset \tau$ is said to be Π^ℓ_m-*unisolvent* if for any real a_ξ, $\xi \in \Xi$, there exists a unique $p \in \Pi^\ell_m(\tau)$ such that $p(\xi) = a_\xi$ for all $\xi \in \Xi$. Obviously, the number of elements in any Π^ℓ_m-unisolvent set is equal to the dimension of Π^ℓ_m.

As a well known example of a Π^ℓ_m-unisolvent set we mention the set of $\binom{\ell+m}{\ell}$ uniformly distributed points in the ℓ-simplex $\tau = \langle v_0, ..., v_\ell \rangle$,

$$\tilde{\Xi}_m(\tau) := \left\{ \xi : \xi = \frac{u_0 v_0 + \cdots + u_\ell v_\ell}{m}, \text{ where } i_0 + \cdots + i_\ell = m \right\}. \tag{2.6}$$

Moreover, its subsets

$$\tilde{\Xi}^k_m(\tau) := \{\xi \in \tilde{\Xi}_m(\tau) : i_j > k,\ j = 0, ..., \ell\}, \qquad 0 \leqslant k \leqslant \frac{m-\ell}{\ell+1}, \tag{2.7}$$

are examples of $\Pi^\ell_{m-(k+1)(\ell+1)}$-unisolvent sets in the *interior* of τ.

The following technical lemma will be very useful later.

LEMMA 2.7. *Let $p \leqslant \Pi^\ell_m(\tau)$ and $0 \leqslant k \leqslant \frac{m-\ell}{\ell+1}$. Suppose that*

(1) *for each facet τ' of τ,*

$$\delta_x D^{k'}_{\sigma(\tau',\tau)} p = 0, \qquad all \quad x \in \tau', \qquad k' = 0, ..., k,$$

(2) *for some $\Pi^\ell_{m-(k+1)(\ell+1)}$-unisolvent set Ξ in the interior of τ,*

$$\delta_\xi p = 0, \qquad all \quad \xi \in \Xi.$$

Then $p = 0$.

Proof. Let $\tau_1, ..., \tau_{\ell+1}$ be all facets of τ. For each τ_i, let p_i be a linear n-variate polynomial such that $p_i|_{\tau_i} = 0$ and $p_i|_\tau \neq 0$. It follows from (1) that

$$p = \tilde{p} \prod_{i=1}^{\ell+1} (p_i|_\tau)^{k+1},$$

where $\tilde{p}$ is a polynomial in $\Pi^\ell_{m-(k+1)(\ell+1)}(\tau)$. Since p_i, $i = 1, ..., \ell+1$, do not vanish in the interior of τ, (2) implies that $\tilde{p}(\xi) = 0$ for all $\xi \in \Xi$. Therefore, $\tilde{p} = 0$, and hence $p = 0$. ∎

3. A NODAL DETERMINING SET FOR $\mathcal{S}^r_d(\Delta)$

Suppose $r \geqslant 1$ and $d \geqslant r2^n + 1$. We now associate with each $\tau \in \mathcal{T}$ a set $\mathcal{N}_\tau$ of nodal functionals on $\mathcal{S}^r_d(\Delta)$. First, let v be a vertex in $\mathcal{V} = \mathcal{T}_0$. For each n-simplex $T \in \Delta$ containing v we define

$$\mathcal{N}_{v,q}(T) := \{\delta_v D^\alpha_{\sigma(v,T)} : \alpha \in \mathbb{Z}^n_+,\ |\alpha| = q\}, \qquad 0 \leqslant q \leqslant r2^{n-1},$$

$$\mathcal{N}_v(T) := \bigcup_{q=0}^{r2^{n-1}} \mathcal{N}_{v,q}(T).$$

Moreover, we set

$$\mathcal{N}_{v,q} := \bigcup_{\substack{T \in \Delta \\ v \in T}} \mathcal{N}_{v,q}(T), \qquad \mathcal{N}_v := \bigcup_{q=0}^{r2^{n-1}} \mathcal{N}_{v,q} = \bigcup_{\substack{T \in \Delta \\ v \in T}} \mathcal{N}_v(T).$$

Suppose now $\tau \in \mathcal{T}_\ell$ for some $\ell \in \{1, ..., n-1\}$. For each $0 \leqslant q \leqslant r2^{n-\ell-1}$, let $\Xi_{\tau,q}$ be a $\Pi^\ell_{\mu_{\ell,q}}$-unisolvent set in the *interior* of τ, where

$$\mu_{\ell,q} := d - q - (r2^{n-\ell} - q + 1)(\ell + 1). \tag{3.1}$$

Given any n-simplex $T \in \Delta$ containing τ, we define for each $\xi \in \Xi_{\tau,q}$,

$$\mathcal{N}_{\tau,q,\xi}(T) := \{\delta_\xi D^\alpha_{\sigma(\tau,T)} : \alpha \in \mathbb{Z}^{n-\ell}_+, |\alpha| = q\}.$$

Moreover, we set

$$\mathcal{N}_\tau(T) := \bigcup_{q=0}^{r2^{n-\ell-1}} \bigcup_{\xi \in \Xi_{\tau,q}} \mathcal{N}_{\tau,q,\xi}(T), \qquad \mathcal{N}_{\tau,q,\xi} := \bigcup_{\substack{T \in \Delta \\ \tau \subset T}} \mathcal{N}_{\tau,q,\xi}(T),$$

$$\mathcal{N}_{\tau,q} := \bigcup_{\xi \in \Xi_{\tau,q}} \mathcal{N}_{\tau,q,\xi}, \qquad \mathcal{N}_\tau := \bigcup_{q=0}^{r2^{n-\ell-1}} \mathcal{N}_{\tau,q} = \bigcup_{\substack{T \in \Delta \\ \tau \subset T}} \mathcal{N}_\tau(T).$$

Finally, for each $T \in \Delta = \mathcal{T}_n$ we define

$$\mathcal{N}_T := \{\delta_\xi : \xi \in \Xi_T\},$$

where Ξ_T is a $\Pi^n_{d-(r+1)(n+1)}$-unisolvent set in the interior of T.

Note that in general the sets $\mathcal{N}_{\tau,q,\xi}(T)$ are not mutually disjoint for different T containing τ. For example, let $\tau = \langle v_0, ..., v_{n-2} \rangle \in \mathcal{T}_{n-2}$, and suppose that both $T = \langle \tau, u, w \rangle$ and $\tilde{T} = \langle \tau, u, \tilde{w} \rangle$ are in Δ. Then the nodal functional $\delta_\xi D^{r+1}_{\sigma_{\tau,u}}$ belongs to $\mathcal{N}_{\tau,r+1,\xi}(T) \cap \mathcal{N}_{\tau,r+1,\xi}(\tilde{T})$. On the other hand, if an n-simplex $T \in \Delta$ is fixed, then the sets $\mathcal{N}_{\tau,q,\xi}(T)$ are mutually disjoint for all τ, q, ξ.

THEOREM 3.1. *The set*

$$\mathcal{N} := \bigcup_{\tau \in \mathcal{T}} \mathcal{N}_\tau$$

is a determining set for $\mathcal{S}^r_d(\Delta)$.

Proof. Let $s \in \mathcal{S}_d^r(\Delta)$ satisfy $\eta s = 0$ for all $\eta \in \mathcal{N}$. We have to show that $s = 0$. To this end we choose an arbitrary $T \in \Delta$ and show that $s|_T = 0$. For each vertex v of T, the set

$$\mathcal{N}_v(T) = \{\delta_v D^\alpha_{\sigma(v,T)} : \alpha \in \mathbb{Z}^n_+,\ |\alpha| \leqslant r2^{n-1}\}$$

is included in $\mathcal{N}$. Since $\sigma(v, T)$ is a basis of $\mathbb{R}^n$, we have by Lemma 2.6,

$$\delta_v D^\alpha_\sigma s|_T = 0, \qquad \text{all} \quad \alpha \in \mathbb{Z}^n_+, \qquad |\alpha| \leqslant r2^{n-1},$$

for any sequence σ of unit vectors.

For $\ell = 0, ..., n-1$, we now show by induction that for each ℓ-face τ of T, if the components of σ are some unit vectors in $(\tau)^\perp$, then

$$\delta_x D^\alpha_\sigma s|_T = 0, \qquad \text{all} \quad x \in \tau, \alpha \in \mathbb{Z}^{n-\ell}_+, \qquad |\alpha| \leqslant r2^{n-\ell-1}. \tag{3.2}$$

The validity of (3.2) for $\ell = 0$ is shown above. Suppose $1 \leqslant \ell \leqslant n-1$. Let $\alpha \in \mathbb{Z}^{n-\ell}_+$, $|\alpha| = q$, with $1 \leqslant q \leqslant r2^{n-\ell-1}$. In view of Lemma 2.6, it suffices to prove (3.2) for $\sigma = \sigma(\tau, T)$. We have $p := D^\alpha_{\sigma(\tau,T)} s|_T \in \Pi^n_{d-q}$ and $p|_\tau \in \Pi^\ell_{d-q}(\tau)$. By the induction hypothesis, for each facet τ' of τ,

$$\delta_x D^{q'}_{\sigma(\tau',\tau)} p|_\tau = 0, \qquad \text{all} \quad x \in \tau', \quad q' = 0, ..., r2^{n-\ell} - q.$$

Since the nodal functionals $\delta_\xi D^\alpha_{\sigma(\tau,T)}$, $\xi \in \Xi_{\tau,q}$, are included in $\mathcal{N}_\tau(T) \subset \mathcal{N}$, we have in addition

$$\delta_\xi p|_\tau = 0, \qquad \text{all} \quad \xi \in \Xi_{\tau,q}.$$

Since $\Xi_{\tau,q}$ is $\Pi^\ell_{\mu_{\ell,q}}$-unisolvent, Lemma 2.7 implies that $p|_\tau = 0$, which confirms (3.2).

In particular, (3.2) holds for each facet F of T, i.e.,

$$\delta_s D^q_{\sigma(F,T)} s|_T = 0, \qquad \text{all} \quad x \in F, \quad q = 0, ..., r.$$

Since $\mathcal{N}_T$ is included in $\mathcal{N}$, we have in addition

$$\delta_\xi s|_T = 0, \qquad \text{all} \quad \xi \in \Xi_T.$$

Since Ξ_T is $\Pi^n_{d-(r+1)(n+1)}$-unisolvent, Lemma 2.7 implies that $s|_T = 0$, which completes the proof. ∎

THEOREM 3.2. *For each $T \in \Delta$, let*

$$\mathcal{N}(T) := \mathcal{N}_T \cup \bigcup_{\ell=0}^{n-1} \bigcup_{\tau \in \mathcal{T}_\ell(T)} \mathcal{N}_\tau(T),$$

where $\mathcal{T}_\ell(T)$ denotes the set of all ℓ-faces of T. Then $\mathcal{N}(T)$ is a minimal determining set for Π_d^n.

Proof. It is easy to see that the set of nodal functionals $\mathcal{N}(T)$ is the same, whatever the triangulation Δ containing T may be. If we take $\Delta = \{T\}$, then obviously $\mathcal{S}_d^r(\Delta) = \Pi_d^n$ and $\mathcal{N} = \mathcal{N}(T)$. Therefore, $\mathcal{N}(T)$ is a determining set for Π_d^n by Theorem 3.1. It thus remains to show that $\#\mathcal{N}(T) = \dim \Pi_d^n = \binom{n+d}{n}$. We have

$$\#\mathcal{N}(T) = \#\mathcal{N}_T + \sum_{v \in \mathcal{T}_0(T)} \#\mathcal{N}_v(T) + \sum_{\ell=1}^{n-1} \sum_{\tau \in \mathcal{T}_\ell(T)} \#\mathcal{N}_\tau(T).$$

It is easy to see that

$$\#\mathcal{N}_T = \binom{n+d-(r+1)(n+1)}{n},$$

$$\#\mathcal{N}_v(T) = \sum_{q=0}^{r2^{n-1}} \binom{n-1+q}{n-1} = \binom{n+r2^{n-1}}{n}, \qquad v \in \mathcal{T}_0(T),$$

$$\#\mathcal{N}_\tau(T) = \sum_{q=0}^{r2^{n-\ell-1}} \binom{\ell+\mu_{\ell,q}}{\ell}\binom{n-\ell-1+q}{n-\ell-1},$$

$$\tau \in \mathcal{T}_\ell(T), \qquad 1 \leqslant \ell \leqslant n-1,$$

where $\mu_{\ell,q}$ is defined in (3.1).

We now consider the set

$$Z := \{\alpha \in \mathbb{Z}_+^{n+1} : |\alpha| = d\}.$$

Obviously, $\#Z = \binom{n+d}{n}$. Therefore, the theorem will be established if we show that

$$\#Z = \#\mathcal{N}(T). \tag{3.3}$$

For any nonempty subset I of $\{1, ..., n+1\}$, let

$$Z_I := \left\{\alpha \in Z : \sum_{i \in I} \alpha_i \geqslant d - r2^{n-\ell-1}\right\}, \qquad \text{if} \quad \ell := \#I - 1 < n,$$

$$Z_{\{1, ..., n+1\}} := Z,$$

and

$$\tilde{Z}_{\{i\}} := Z_{\{i\}}, \qquad i = 1, ..., n+1,$$

$$\tilde{Z}_I := Z_I \Big\backslash \bigcup_{i \in I} Z_{I \backslash \{i\}}, \qquad \# I \geqslant 2.$$

Taking into account the assumption $d \geqslant r2^n + 1$, it is not difficult to see that Z is a disjoint union of the sets $\tilde{Z}_I$. Hence,

$$\# Z = \sum_{\ell=0}^{n} \sum_{\# I = \ell + 1} \# \tilde{Z}_I.$$

We have

$$\tilde{Z}_{\{1, ..., n+1\}} = \left\{ \alpha \in Z : \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \alpha_i < d - r, \; j = 1, ..., n+1 \right\}$$
$$= \{ \alpha \in \mathbb{Z}_+^{n+1} : |\alpha| = d, \; \alpha_j \geqslant r + 1, \; j = 1, ..., n+1 \},$$

and it follows that

$$\# \tilde{Z}_{\{1, ..., n+1\}} = \binom{n + d - (r+1)(n+1)}{n} = \# \mathcal{N}_T.$$

Furthermore, for each $i = 1, ..., n+1$, we have

$$\tilde{Z}_{\{i\}} = \{ \alpha \in \mathbb{Z}_+^{n+1} : |\alpha| = d, \; \alpha_i \geqslant d - r2^{n-1} \},$$

so that $\# \tilde{Z}_{\{i\}} = \binom{n + r2^{n-1}}{n}$, and hence

$$\sum_{i=1}^{n+1} \# \tilde{Z}_{\{i\}} = (n+1) \binom{n + r2^{n-1}}{n} = \sum_{v \in \mathcal{T}_0(T)} \# \mathcal{N}_v(T).$$

Let now $I \subset \{1, ..., n+1\}$, $\ell := \# I - 1 < n$. Then

$$\tilde{Z}_I = \left\{ \alpha \in Z : \sum_{i \in I} \alpha_i \geqslant d - r2^{n-\ell-1}, \; \sum_{i \in I \backslash \{j\}} \alpha_i < d - r2^{n-\ell}, \; j \in I \right\}$$
$$= \bigcup_{q=0}^{r2^{n-\ell-1}} \left\{ \alpha \in Z : \sum_{i \in I} \alpha_i = d - q, \; \alpha_j \geqslant r2^{n-\ell} - q + 1, \; j \in I \right\}.$$

A standard combinatorial argument shows that the cardinality of the set

$$\left\{\alpha \in Z : \sum_{i \in I} \alpha_i = d - q,\ \alpha_j \geqslant r2^{n-\ell} - q + 1,\ j \in I\right\}$$

is $\binom{\ell + \mu_{\ell,q}}{\ell}\binom{n-\ell-1+q}{n-\ell-1}$. Since the number of subsets I of $\{1, ..., n+1\}$ consisting of $\ell + 1$ elements is equal to $\binom{n+1}{\ell+1} = \# \mathcal{T}_\ell(T)$, we conclude that

$$\sum_{\# I = \ell+1} \# \tilde{Z}_I = \sum_{\tau \in \mathcal{T}_\ell(T)} \# \mathcal{N}_\tau(T), \qquad \ell = 1, ..., n-1.$$

Thus, (3.3) holds, and the proof is complete. ∎

Theorem 3.2 shows that the set $\mathcal{N}(T)$ defines a *Hermite interpolation operator* $\mathcal{H}_T : C^{r2^{n-1}}(T) \to \Pi_d^n$ as follows. Given $f \in C^{r2^{n-1}}(T)$, let $\mathcal{H}_T f$ be the unique polynomial in Π_d^n satisfying

$$\eta \mathcal{H}_T f = \eta f, \qquad \text{all} \quad \eta \in \mathcal{N}(T). \tag{3.4}$$

Obviously, this is a standard finite-element interpolation scheme, see e.g. [24, 30].

The following estimation of the norm of $\mathcal{H}_T f$ in the case of uniformly distributed points easily follows from the general results given in [13]; see also the proof of Lemma 3.9 in [16].

LEMMA 3.3. *Choose*

$$\begin{aligned} \Xi_{\tau,q} &= \tilde{\Xi}_{d-q}^{r2^{n-\ell}-q}, && \textit{all} \quad \tau \in \mathcal{T}_\ell,\ 1 \leqslant \ell \leqslant n-1,\ 0 \leqslant q \leqslant r2^{n-\ell-1}, \\ \Xi_T &= \tilde{\Xi}_d^r, && \textit{all} \quad T \in \mathcal{T}_n, \end{aligned} \tag{3.5}$$

where $\tilde{\Xi}_m^k$ *are defined in* (2.7). *Then*

$$\| \mathcal{H}_T f \|_{L_\infty(T)} \leqslant K \max_{\eta \in \mathcal{N}(T)} h_T^{q(\eta)} |\eta f|, \tag{3.6}$$

where h_T is the diameter of T, $q(\eta)$ is the order of the nodal functional η, and K is a constant depending only on n, r and d.

4. SMOOTHNESS CONDITIONS

As shown in the previous section, $\mathcal{N} \subset \mathcal{S}_d^r(\Delta)^*$ is a determining set for $\mathcal{S}_d^r(\Delta)$. Therefore, $\mathcal{N}$ is a spanning set for $\mathcal{S}_d^r(\Delta)^*$. However, as we will see, there are some linear dependencies between the elements of $\mathcal{N}$, called *nodal smoothness conditions*. Our next task is to describe these conditions.

Let $\tau \in \mathcal{T}_\ell$ for some $0 \leqslant \ell \leqslant n-1$, and let $F = \langle \tau, u_1, ..., u_{n-\ell-1} \rangle \in \mathcal{T}_{n-1}$ be an *interior* facet of Δ attached to τ. Then there are exactly two n-simplices T_1, $T_2 \in \Delta$ sharing the facet F. Let $T_1 = \langle F, u_{n-\ell} \rangle$, $T_2 = \langle F, w \rangle$. Since the components of

$$\sigma(\tau, T_1) = (\sigma_{\tau, u_1}, ..., \sigma_{\tau, u_{n-\ell}})$$

form a basis for $(\tau)^\perp$, and since $\sigma_{\tau, w}$ also lies in $(\tau)^\perp$, there exists $\mu \in \mathbb{R}^{n-\ell}$, $\mu = (\mu_1, ..., \mu_{n-\ell})$, such that

$$\sigma_{\tau, w} = \sum_{i=1}^{n-\ell} \mu_i \sigma_{\tau, u_i}.$$

LEMMA 4.1. *If $s \in \mathcal{S}_d^r(\Delta)$, then for all $\xi \in \tau$, $\alpha \in \mathbb{Z}_+^{n-\ell-1}$ and $0 \leqslant r' \leqslant r$,*

$$\delta_\xi \, D^\alpha_{\sigma(\tau, F)} \, D^{r'}_{\sigma_{\tau, w}} s = \sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \, \delta_\xi \, D^\alpha_{\sigma(\tau, F)} \, D^\beta_{\sigma(\tau, T_1)} s, \tag{4.1}$$

where $\binom{|\beta|}{\beta} := |\beta|!/\beta_1! \cdots \beta_{n-\ell}!$, $\mu^\beta := \mu_1^{\beta_1} \cdots \mu_{n-\ell}^{\beta_{n-\ell}}$.

Proof. Let $p_1 := s|_{T_1}$, $p_2 := s|_{T_2}$ and $\sigma_i := \sigma_{\tau, u_i}$, $i = 1, ..., n-\ell$. We have

$$\begin{aligned} \delta_\xi \, D^\alpha_{\sigma(\tau, F)} \, D^{r'}_{\sigma_{\tau, w}} p_1 &= \delta_\xi \, D^\alpha_{\sigma(\tau, F)} \left(\sum_{i=1}^{n-\ell} \mu_i \, D_{\sigma_i} \right)^{r'} p_1 \\ &= \delta_\xi \, D^\alpha_{\sigma(\tau, F)} \left(\sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \, D^{\beta_1}_{\sigma_1} \cdots D^{\beta_{n-\ell}}_{\sigma_{n-\ell}} \right) p_1 \\ &= \sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \, \delta_\xi \, D^\alpha_{\sigma(\tau, F)} \, D^\beta_{\sigma(\tau, T_1)} p_1. \end{aligned}$$

Since $s \in C^r(T_1 \cup T_2)$ and $r' \leqslant r$,

$$D^{r'}_{\sigma_{\tau, w}} p_1(x) = D^{r'}_{\sigma_{\tau, w}} p_2(x), \qquad \text{all} \quad x \in F = T_1 \cap T_2.$$

Therefore,

$$\delta_\xi \, D^\alpha_{\sigma(\tau, F)} \, D^{r'}_{\sigma_{\tau, w}} p_1 = \delta_\xi \, D^\alpha_{\sigma(\tau, F)} \, D^{r'}_{\sigma_{\tau, w}} p_2,$$

for all $\xi \in F$, in particular for $\xi \in \tau$. Thus,

$$\delta_\xi \, D^\alpha_{\sigma(\tau, F)} \, D^{r'}_{\sigma_{\tau, w}} p_2 = \sum_{\substack{\beta \in \mathbb{Z}_+^{n-\ell} \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \, \delta_\xi \, D^\alpha_{\sigma(\tau, F)} \, D^\beta_{\sigma(\tau, T_1)} p_1. \tag{4.2}$$

Finally, we note that

$$D^{\alpha}_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} = D^{\gamma}_{\sigma(\tau,T_2)}, \qquad D^{\alpha}_{\sigma(\tau,F)} D^{\beta}_{\sigma(\tau,T_1)} = D^{\tilde{\gamma}}_{\sigma(\tau,T_1)}, \tag{4.3}$$

where $\gamma = (\alpha_1, ..., \alpha_{n-\ell-1}, r')$, $\tilde{\gamma} = (\alpha_1 + \beta_1, ..., \alpha_{n-\ell-1} + \beta_{n-\ell-1}, \beta_{n-\ell})$, and the observation that by definition

$$\delta_\xi D^{\gamma}_{\sigma(\tau,T_2)} s = \delta_\xi D^{\gamma}_{\sigma(\tau,T_2)} p_2, \qquad \delta_\xi D^{\tilde{\gamma}}_{\sigma(\tau,T_1)} s = \delta_\xi D^{\tilde{\gamma}}_{\sigma(\tau,T_1)} p_1$$

(see Section 2.3) completes the proof. ∎

Remark 4.2. Lemma 4.1 shows that the condition (4.2) holds for all $\xi \in \tau$, $\alpha \in \mathbb{Z}^{n-\ell}_{+}$ and $0 \leqslant r' \leqslant r$ if the two polynomials p_1 and p_2 defined on T_1 and T_2, respectively, join together with C^r-smoothness across $F = T_1 \cap T_2$. It is not difficult to see that the converse is also true. Note that for $\tau \in \mathcal{T}_0$, Lemma 4.1 as well as its converse were given (in a slightly different form) in Theorem 4.1.2 of [11], and (in the bivariate case) in [16].

We now concentrate on the conditions (4.1) that involve the nodal functionals in the set $\mathcal{N}$ defined in Section 3. Namely, Lemma 4.1 implies that the following linear relations between the elements of $\mathcal{N}$ hold:

(1) given $v \in \mathcal{T}_0$ and $0 \leqslant q \leqslant r2^{n-1}$, the system $\mathcal{R}_{v,q}$ of linear conditions

$$\delta_v D^{\alpha}_{\sigma(v,F)} D^{r'}_{\sigma_{v,w}} = \sum_{\substack{\beta \in \mathbb{Z}^n_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \delta_v D^{\alpha}_{\sigma(v,F)} D^{\beta}_{\sigma(v,T_1)}, \tag{4.4}$$

for all $0 \leqslant r' \leqslant \min\{r, q\}$, all $\alpha \in \mathbb{Z}^{n-1}_{+}$, with $|\alpha| = q - r'$, and all interior facets $F \in \mathcal{T}_{n-1}$ such that $v \in F$,

(2) given $\tau \in \mathcal{T}_\ell$ (where $1 \leqslant \ell \leqslant n-2$), $0 \leqslant q \leqslant r2^{n-\ell-1}$, and $\xi \in \Xi_{\tau,q}$, the system $\mathcal{R}_{\tau,q,\xi}$ of linear conditions

$$\delta_\xi D^{\alpha}_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}} = \sum_{\substack{\beta \in \mathbb{Z}^{n-\ell}_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \delta_\xi D^{\alpha}_{\sigma(\tau,F)} D^{\beta}_{\sigma(\tau,T_1)}, \tag{4.5}$$

for all $0 \leqslant r' \leqslant \min\{r, q\}$, all $\alpha \in \mathbb{Z}^{n-\ell-1}_{+}$, with $|\alpha| = q - r'$, and all interior facets $F \in \mathcal{T}_{n-1}$ such that $\tau \subset F$, and

(3) given an interior facet $F \in \mathcal{T}_{n-1}$, $0 \leqslant q \leqslant r$, and $\xi \in \Xi_{F,q}$, the linear condition $\mathcal{R}_{F,q,\xi}$,

$$\delta_\xi D^{q}_{\sigma_{F,w}} = (-1)^q \delta_\xi D^{q}_{\sigma(F,T_1)}. \tag{4.6}$$

(Here and above w, T_1 and μ_i correspond to a particular F and are defined as in Lemma 4.1.)

Remark 4.3. In view of (4.3) it is easy to see that the smoothness conditions in $\mathcal{R}_{v,q}$, $\mathcal{R}_{\tau,q,\xi}$ or $\mathcal{R}_{F,q,\xi}$ involve only the nodal functionals in $\mathcal{N}_{v,q}$, $\mathcal{N}_{\tau,q,\xi}$ or $\mathcal{N}_{F,q,\xi}$, respectively. (See the definition of the sets of nodal functionals $\mathcal{N}_{v,q}$ and $\mathcal{N}_{\tau,q,\xi}$ in Section 3.)

Let

$$\mathcal{R}_v := \bigcup_{q=0}^{r2^{n-1}} \mathcal{R}_{v,q}, \qquad v \in \mathcal{T}_0,$$

$$\mathcal{R}_\tau := \bigcup_{q=0}^{r2^{n-\ell-1}} \mathcal{R}_{\tau,q} \qquad \mathcal{R}_{\tau,q} := \bigcup_{\xi \in \Xi_{\tau,q}} \mathcal{R}_{\tau,q,\xi}, \qquad \tau \in \mathcal{T}_\ell, \qquad 1 \leqslant \ell \leqslant n-1. \tag{4.7}$$

THEOREM 4.4. *The set*

$$\mathcal{R} := \bigcup_{\tau \in \mathcal{T} \setminus \mathcal{T}_n} \mathcal{R}_\tau \tag{4.8}$$

is a complete system of linear relations for $\mathcal{N}$ *over* $\mathcal{S}^r_d(\Delta)$.

Proof. By Theorem 3.1, $\mathcal{N}$ is a determining set for $\mathcal{S}^r_d(\Delta)$. Suppose the system $\mathcal{R}$ is written as

$$\sum_{j \in J} c_{i,j} \eta_j = 0, \qquad i \in I,$$

where I, J are some index sets, $\{\eta_j\}_{j \in J} = \mathcal{N}$, and $c_{i,j}$ real coefficients. Let a_j, $j \in J$, be any real numbers satisfying

$$\sum_{j \in J} c_{i,j} a_j = 0, \qquad i \in I.$$

According to Definition 2.2, we have to show that there exists a spline $s \in \mathcal{S}^r_d(\Delta)$ such that $\eta_j s = a_j$ for all $j \in J$. We first construct the polynomial pieces of s, $p_T = s|_T$, $T \in \Delta$, as follows. By Theorem 3.2, $\mathcal{N}(T)$ is a minimal determining set for Π^n_d. We define p_T to be the unique polynomial in Π^n_d such that

$$\eta_j p_T = a_j, \qquad \text{all} \quad \eta_j \in \mathcal{N}(T).$$

We thus have to prove that p_T, $T \in \Delta$, join together with C^r-smoothness. To this end it suffices to consider two n-simplices $T_1, T_2 \in \Delta$ sharing a facet

$F \in \mathscr{T}_{n-1}$ and show that the two polynomials $p_1 := p_{T_1}$ and $p_2 := p_{T_2}$ join with C^r-smoothness across F. This, in turn, will follow if we show that

$$\delta_x D^{r'}_{\sigma_{F,w}}(p_2 - p_1) = 0, \qquad \text{all} \quad x \in F, \quad r' = 0, ..., r. \tag{4.9}$$

where w is the vertex of T_2 not lying in F. (That is, $T_2 = \langle F, w \rangle$.)

We first prove by induction on ℓ that for each ℓ-face τ of F, $\ell = 0, ..., n-2$, and for all $r' = 0, ..., r$, and $\alpha \in \mathbb{Z}^{n-\ell-1}$, with $|\alpha| \leqslant r2^{n-\ell-1} - r'$,

$$\delta_x D^{\alpha}_{\sigma(\tau, F)} D^{r'}_{\sigma_{\tau,w}}(p_2 - p_1) = 0, \qquad \text{all} \quad x \in \tau. \tag{4.10}$$

Let $\ell = 0$, and let v be a vertex of F. Given $r' = 0, ..., r$ and $\alpha \in \mathbb{Z}^{n-1}$, with $|\alpha| \leqslant r2^{n-1} - r'$, the functional $\eta_{j_0} := \delta_v D^{\alpha}_{\sigma(v,F)} D^{r'}_{\sigma_{v,w}}$ is in $\mathscr{N}(T_2)$. Hence, $\eta_{j_0} p_2 = a_{j_0}$. Let us compute $\eta_{j_0} p_1$. We set $\eta_{j_\beta} := \delta_v D^{\alpha}_{\sigma(v,F)} D^{\beta}_{\sigma(v,T_1)} \in \mathscr{N}(T_1)$, $|\beta| = r'$. By (4.4), the equation

$$\eta_{j_0} - \sum_{\substack{\beta \in \mathbb{Z}^n_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \eta_{j_\beta} = 0$$

belongs to $\mathscr{R}$. Therefore,

$$a_{j_0} - \sum_{\substack{\beta \in \mathbb{Z}^n_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta a_{j_\beta} = 0.$$

On the other hand, since $\eta_{j_\beta} \in \mathscr{N}(T_1)$, we have $\eta_{j_\beta} p_1 = a_{j_\beta}$, and it follows that

$$\eta_{j_0} p_1 = \sum_{\substack{\beta \in \mathbb{Z}^n_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \eta_{j_\beta} p_1 = \sum_{\substack{\beta \in \mathbb{Z}^n_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta a_{j_\beta} = a_{j_0}.$$

Thus, $\eta_{j_0}(p_2 - p_1) = 0$, which confirms (4.10) for $\ell = 0$.

Suppose $1 \leqslant \ell \leqslant n-2$, and let τ be and ℓ-face of F. Given $r' = 0, ..., r$ and $\alpha \in \mathbb{Z}^{n-\ell-1}$, with $|\alpha| \leqslant r2^{n-\ell-1} - r'$, consider

$$p := D^{\alpha}_{\sigma(\tau,F)} D^{r'}_{\sigma_{\tau,w}}(p_2 - p_1)|_\tau \in \Pi^\ell_{d-q}(\tau),$$

where $q := |\alpha| + r'$. Let us show that for each facet τ' of τ,

$$\delta_x D^{q'}_{\sigma(\tau',\tau)} p = 0, \qquad \text{all} \quad x \in \tau', \quad q' = 0, ..., r2^{n-\ell} - q. \tag{4.11}$$

Since the components of $\sigma(\tau', \tau)$ and $\sigma(\tau, F)$ form a basis for $(\tau')^{\perp} \cap (F)$, we have by Lemma 2.6, that

$$D^{q'}_{\sigma(\tau', \tau)} D^{\alpha}_{\sigma(\tau, F)} = \sum_{\substack{\gamma \in \mathbb{Z}^{n-\ell} \\ |\gamma| = |\alpha| + q'}} c_\gamma D^{\gamma}_{\sigma(\tau', F)}.$$

Moreover, since $\sigma_{\tau, w} \in (\tau)^{\perp} \subset (\tau')^{\perp}$,

$$D^{r'}_{\sigma_{\tau, w}} = \sum_{\tilde{r}=0}^{r'} \sum_{\substack{\gamma \in \mathbb{Z}^{n-\ell} \\ |\gamma| = r' - \tilde{r}}} \tilde{c}_{\gamma, \tilde{r}} D^{\gamma}_{\sigma(\tau', F)} D^{\tilde{r}}_{\sigma_{\tau', w}}.$$

Therefore, we have for $x \in \tau'$,

$$\begin{aligned} \delta_x D^{q'}_{\sigma(\tau', \tau)} p &= \delta_x D^{q'}_{\sigma(\tau', \tau)} D^{\alpha}_{\sigma(\tau, F)} D^{r'}_{\sigma_{\tau, w}} (p_2 - p_1) \\ &= \sum_{\tilde{r}=0}^{r'} \sum_{\substack{\gamma \in \mathbb{Z}^{n-\ell} \\ |\gamma| = |\alpha| + q'}} \sum_{\substack{\tilde{\gamma} \in \mathbb{Z}^{n-\ell} \\ |\tilde{\gamma}| = r' - \tilde{r}}} c_\gamma \tilde{c}_{\tilde{\gamma}, \tilde{r}} \delta_x D^{\gamma + \tilde{\gamma}}_{\sigma(\tau', F)} D^{\tilde{r}}_{\sigma_{\tau', w}} (p_2 - p_1). \end{aligned}$$

By the induction hypothesis, every term in this last sum is zero (since $\tilde{r} \leqslant r$ and $|\gamma| + |\tilde{\gamma}| + \tilde{r} = |\alpha| + q' + r' = q + q' \leqslant r 2^{n-\ell}$), and (4.11) follows. We show now that

$$\delta_\xi p = 0, \qquad \text{all} \quad \xi \in \Xi_{\tau, q}, \tag{4.12}$$

where $\Xi_{\tau, q}$ is a $\Pi^{\ell}_{\mu_{\ell, q}}$-unisolvent set in the interior of τ as defined in Section 3. Let $\xi \in \Xi_{\tau, q}$ be given. Similar to the proof in case $\ell = 0$, we set $\eta_{j_0} := \delta_\xi D^{\alpha}_{\sigma(\tau, F)} D^{r'}_{\sigma_{\tau, w}} \in \mathcal{N}(T_2)$, $\eta_{j_\beta} := \delta_\xi D^{\alpha}_{\sigma(\tau, f)} D^{\alpha}_{\sigma(\tau, T_1)} \in \mathcal{N}(T_1)$, $|\beta| = r'$. By (4.5), the equation

$$\eta_{j_0} - \sum_{\substack{\beta \in \mathbb{Z}^{n-\ell}_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \eta_{j_\beta} = 0$$

belongs to $\mathcal{R}$. Hence, we get

$$\begin{aligned} \eta_{j_0} p_1 &= \sum_{\substack{\beta \in \mathbb{Z}^{n-\ell}_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta \eta_{j_\beta} p_1 = \sum_{\substack{\beta \in \mathbb{Z}^{n-\ell}_+ \\ |\beta| = r'}} \binom{|\beta|}{\beta} \mu^\beta a_{j_\beta} \\ &= a_{j_0} = \eta_{j_0} p_2, \end{aligned}$$

and (4.12) is proved. In view of (4.11) and (4.12), we conclude by Lemma 2.7 that $p = 0$, which establishes (4.10).

To prove (4.9) for any given $r' = 0, ..., r$, we set

$$p := D^{r'}_{\sigma_{F,w}}(p_2 - p_1)|_F \in \Pi^{n-1}_{d-r'}.$$

Analysis similar to the above shows that by (4.10) it follows that for each facet τ of F,

$$\delta_x D^q_{\sigma(\tau,F)} p = 0, \qquad \text{all} \quad x \in \tau, q = 0, ..., 2r - r'.$$

Furthermore, given $\xi \in \Xi_{F,x'}$, the nodal functionals $\eta_{j_1} := \delta_\xi D^{r'}_{\sigma(F,T_1)}$ and $\eta_{j_2} := \delta_\xi D^{r'}_{\sigma_{F,w}}$ are in $\mathcal{N}(T_1)$ and $\mathcal{N}(T_2)$, respectively. By (4.6),

$$\delta_\xi D^{r'}_{\sigma_{F,w}} = (-1)^{r'} \delta_\xi D^{r'}_{\sigma(F,T_1)},$$

and hence

$$\delta_\xi p = \eta_{j_2} p_2 - (-1)^{r'} \eta_{j_1} p_1 = a_{j_2} - (-1)^{r'} a_{j_1} = 0.$$

Thus, Lemma 2.7 implies that $p = 0$, which establishes (4.9) and completes the proof of the theorem. ∎

5. CONSTRUCTION OF A LOCAL BASIS FOR $\mathcal{S}^r_d(\Delta)$

Let $d \geq r2^n + 1$. Since $\mathcal{N}$ is a determining set for $\mathcal{S}^r_d(\Delta)$ by Theorem 3.1, and $\mathcal{R}$ is a complete system of linear relations for $\mathcal{N}$ over $\mathcal{S}^r_d(\Delta)$ by Theorem 4.4, Algorithm 2.4 can be applied to construct a basis $\{\tilde{s}_1, ..., \tilde{s}_m\}$ for $\mathcal{S}^r_d(\Delta)$. To this end we only need to choose a basis $\{a^{[1]}, ..., a^{[m]}\}$ for the null space $N(C)$ of the corresponding matrix C. In this section we will show how to choose the basis for $N(C)$ so that the resulting basis for $\mathcal{S}^r_d(\Delta)$ is *local* as defined below.

Let v be a vertex of Δ. We set $\text{star}^1(v) := \text{star}(v)$, and define $\text{star}^\gamma(v)$, $\gamma \geq 2$, recursively as the union of the stars of the vertices in $\mathcal{T}_0 \cap \text{star}^{\gamma-1}(v)$.

DEFINITION 5.1. Let $\mathcal{S}$ be a linear subspace of $\mathcal{S}^r_d(\Delta)$. A basis $\{s_1, ..., s_m\}$ for $\mathcal{S}$ is called *local* (or γ-*local*) if there is an integer γ such that for each $k = 1, ..., m$, $\text{supp}\, s_k \subset \text{star}^\gamma(v_k)$, for some vertex v_k of Δ, and the dual functionals $\lambda_1, ..., \lambda_m$, defined by (2.1), can be localized in the same sets $\text{star}^\gamma(v_1), ..., \text{star}^\gamma(v_k)$, i.e., for each $k = 1, ..., m$, $\lambda_k s = 0$ for all $s \in \mathcal{S}$ satisfying $s|_{\text{star}^\gamma(v_k)} = 0$.

We say that an *algorithm produces local bases* if there exists an absolute (integer) constant γ such that any basis constructed by that algorithm is at most γ-local.

The key observation for our construction is that the matrix C of the system $\mathscr{R}$ has a *block diagonal structure*. More precisely, by Remark 4.3 we have

$$C = [\,\tilde{C}\; O\,], \qquad (5.1)$$
$$\tilde{C} = \operatorname{diag}(C_\tau)_{\tau \in \mathscr{T} \setminus \mathscr{T}_n},$$

where C_τ is the matrix of the system $\mathscr{R}_\tau$ defined in (4.7), and O is the zero matrix corresponding to the nodal functionals in $\mathscr{N}_T$, $T \in \mathscr{T}_n$, not involved in any smoothness conditions. Moreover, each matrix C_τ itself is block diagonal. Namely,

$$C_\tau = \operatorname{diag}(C_{\tau,q})_{q=0,\ldots,r2^{n-\ell-1}}, \qquad \tau \in \mathscr{T}_\ell, \qquad 0 \leqslant \ell \leqslant n-1, \qquad (5.2)$$

where $C_{\tau,q}$ is the matrix of the system $\mathscr{R}_{\tau,q}$ defined in (4.4)–(4.7). If $1 \leqslant \ell \leqslant n-1$, then the matrix $C_{\tau,q}$ is again block diagonal,

$$C_{\tau,q} = \operatorname{diag}(C_{\tau,q,\xi})_{\xi \in \Xi_{\tau,q}},$$

with $C_{\tau,q,\xi}$ being the matrix of the system $\mathscr{R}_{\tau,q,\xi}$. By Lemma 2.3, we have

$$\begin{aligned} \dim \mathscr{S}^r_d(\Delta) &= \#\mathscr{N} - \sum_{\tau \in \mathscr{T} \setminus \mathscr{T}_n} \operatorname{rank} C_\tau \\ &= \#\mathscr{N} - \sum_{v \in \mathscr{T}_0} \sum_{q=0}^{r2^{n-1}} \operatorname{rank} C_{v,q} \\ &\quad - \sum_{\ell=1}^{n-1} \sum_{\tau \in \mathscr{T}_\ell} \sum_{q=0}^{r2^{n-\ell-1}} \sum_{\xi \in \Xi_{\tau,q}} \operatorname{rank} C_{\tau,q,\xi}. \end{aligned} \qquad (5.3)$$

Remark 5.2. The formula (5.3) leads to the efficient computation of the dimension of the space $\mathscr{S}^r_d(\Delta)$ by applying to the *small* matrices $C_{v,q}$ and $C_{\tau,q,\xi}$ the standard numerical algorithms of rank determination (see e.g. [29]).

In view of (5.1) and (5.2), $N(\tilde{C})$ is an (outer) direct sum of $N(C_{\tau,q})$, $q = 0, \ldots, r2^{n-\ell-1}$, $\tau \in \mathscr{T}_\ell$, $0 \leqslant \ell \leqslant n-1$. Hence, if we know bases for all $N(C_{\tau,q})$, then we can combine them into a basis for $N(\tilde{C})$ that trivially extends to a basis for $N(C)$. Let $\mathscr{N}_{\tau,q} = \{\eta_j^{[\tau,q]}\}_{j \in J_{\tau,q}}$ and $C_{\tau,q} = (c_{i,j}^{[\tau,q]})_{i \in I_{\tau,q},\, j \in J_{\tau,q}}$, so that $\mathscr{R}_{\tau,q}$ has the form

$$\sum_{j \in J_{\tau,q}} c_{i,j}^{[\tau,q]} \eta_j^{[\tau,q]} = 0, \qquad i \in I_{\tau,q}.$$

For each $\tau \in \mathcal{T}_\ell$, $0 \leqslant \ell \leqslant n-1$, and $q = 0, ..., r2^{n-\ell-1}$, suppose

$$a^{[\tau, q, k]} = (a_j^{[\tau, q, k]})_{j \in J_{\tau, q}}, \qquad k = 1, ..., m_{\tau, q}, \tag{5.4}$$

form a basis for $N(C_{\tau, q})$. In addition, for each $T \in \mathcal{T}_n$, let $a^{[T, 0, k]} = (a_j^{[T, 0, k]})_{j \in J_{T, 0}}$, $k = 1, ..., m_T$, be any basis of $\mathbb{R}^{m_T}$, where $m_T = \# J_{T, 0} = \# \mathcal{N}_T = \# \Xi_T$. We define $\tilde{a}^{[\tau, q, k]} = (\tilde{a}_j^{[\tau, q, k]})_{j \in J}$, with $J = \bigcup_{\tau, q} J_{\tau, q}$, by

$$\tilde{a}_j^{[\tau, q, k]} := \begin{cases} a_j^{[\tau, q, k]}, & \text{if} \quad j \in J_{\tau, q}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the vectors $\tilde{a}^{[\tau, q, k]}$, $k = 1, ..., m_{\tau, q}$, $q = 0, ..., q_\ell$, $\tau \in \mathcal{T}_\ell$, $0 \leqslant \ell \leqslant n$, where

$$q_\ell = \begin{cases} r2^{n-\ell-1}, & \text{if} \quad 0 \leqslant \ell \leqslant n-1, \\ 0, & \text{if} \quad \ell = n, \end{cases} \tag{5.5}$$

obviously form a basis for $N(C)$. The corresponding basis

$$\tilde{s}^{[\tau, q, k]}, \qquad k = 1, ..., m_{\tau, q}, \qquad q = 0, ..., q_\ell, \qquad \tau \in \mathcal{T}_\ell, \qquad 0 \leqslant \ell \leqslant n, \tag{5.6}$$

for $\mathcal{S}_d^r(\Delta)$ produced by Algorithm 2.4 satisfies

$$\begin{aligned} \eta_j^{[\tau, q]} \tilde{s}^{[\tau, q, k]} &= a_j^{[\tau, q, k]}, & j &\in J_{\tau, q}, \\ \eta \tilde{s}^{[\tau, q, k]} &= 0, & \text{all} \quad \eta &\in \mathcal{N} \backslash \mathcal{N}_{\tau, q}. \end{aligned} \tag{5.7}$$

Denote by

$$\tilde{\lambda}^{[\tau, q, k]}, \qquad k = 1, ..., m_{\tau, q}, \qquad q = 0, ..., q_\ell, \qquad \tau \in \mathcal{T}_\ell, \qquad 0 \leqslant \ell \leqslant n, \tag{5.8}$$

the dual basis for $\mathcal{S}_d^r(\Delta)^*$ determined by the duality condition

$$\tilde{\lambda}^{[\tau, q, k]} \tilde{s}^{[\tau', q', k']} = \begin{cases} 1, & \text{if} \quad \tau = \tau', q = q' \quad \text{and} \quad k = k', \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 5.3. *The basis* (5.6) *for* $\mathcal{S}_d^r(\Delta)$, *where* $d \geqslant r2^n + 1$, *is local. Moreover,*

$$\operatorname{supp} \tilde{s}^{[\tau, q, k]} \subset \operatorname{star}(\tau), \tag{5.9}$$

and the dual basis (5.8) *satisfies*

$$\tilde{\lambda}^{[\tau, q, k]} s = 0 \qquad \textit{for all} \quad s \in \mathcal{S}_d^r(\Delta) \qquad \textit{such that} \quad s|_{\operatorname{star}(\tau)} = 0. \tag{5.10}$$

Proof. By (5.7) we have $\eta\tilde{s}^{[\tau, q, k]} = 0$ for all $\eta \in \mathcal{N} \setminus \mathcal{N}_{\tau, q}$. Since $\mathcal{N}_{\tau, q} \cap \mathcal{N}(T) \neq \emptyset$ only if $\tau \subset T$, (5.9) follows from the fact that $\mathcal{N}(T)$ is a determining set for Π^n_d, see Theorem 3.2. To show (5.10), we consider the matrix A with columns

$$\tilde{a}^{[\tau, q, k]}, \qquad k = 1, ..., m_{\tau, q}, \qquad q = 0, ..., q_\ell, \qquad \tau \in \mathcal{T}_\ell, \qquad 0 \leqslant \ell \leqslant n.$$

This matrix is block diagonal,

$$A = \operatorname{diag}(A_\tau)_{\tau \in \mathcal{T}},$$

$$A_\tau = \operatorname{diag}(A_{\tau, q})_{q = 0, ..., q_\ell}, \qquad \tau \in \mathcal{T}_\ell, \qquad 0 \leqslant \ell \leqslant n,$$

where $A_{\tau, q} := (a_j^{[\tau, q, k]})_{j \in J_{\tau, q}, k = 1, ..., m_{\tau, q}}$. Let $B_{\tau, q}$ be a left inverse of $A_{\tau, q}$. Then $B := \operatorname{diag}(B_\tau)_{\tau \in \mathcal{T}}$, with $B_\tau = \operatorname{diag}(B_{\tau, q})_{q = 0, ..., q_\ell}$, $\tau \in \mathcal{T}_\ell$, $0 \leqslant \ell \leqslant n$, is a left inverse of A. Hence, by Lemma 2.5, $\tilde{\lambda}^{[\tau, q, k]}$ is a linear combination of $\eta_j^{[\tau, q]}$, $j \in J_{\tau, q}$. This implies (5.10) since for every $\eta \in \mathcal{N}_{\tau, q}$ we obviously have $\eta s = 0$ if $s|_{\operatorname{star}(\tau)} = 0$. ∎

Remark 5.4. A similar analysis of the space $\mathcal{S}^r_d(\Delta)$, $d \geqslant r2^n + 1$, was done in [2] by using Bernstein–Bézier smoothness conditions [5]. However, the existence of a local basis for $\mathcal{S}^r_d(\Delta)$ was shown in [2] only for $n \leqslant 3$. The main advantage of the nodal techniques used here is that the matrix $\tilde{C}$ in (5.1) is block diagonal, while the matrix of Bernstein–Bézier smoothness conditions is block triangular (see [6]).

6. A STABLE LOCAL BASIS FOR $\mathcal{S}^r_d(\Delta)$

In this section we show that if the sets $\Xi_{\tau, q}$ and Ξ_T as well as the bases (5.4) for $N(C_{\tau, q})$ are properly chosen, then an appropriately renormalized version of the local basis for $S^r_d(\Delta)$ constructed above is in addition stable.

Let us denote by ω_Δ the *shape regularity constant* of the triangulation Δ,

$$\omega_\Delta := \max_{T \in \Delta} \frac{h_T}{\rho_T},$$

where h_T and ρ_T are the diameter of T and the diameter of its inscribed sphere, respectively. Given $M = \bigcup_{T \in \tilde{\Delta}} T$, where $\tilde{\Delta} \subset \Delta$, we denote by $|M|$ the *n-dimensional volume* of M.

DEFINITION 6.1. Let $\mathscr{S}$ be a linear subspace of $\mathscr{S}_d^r(\Delta)$. We say that a basis $\{\tilde{s}_1, ..., \tilde{s}_m\}$ for $\mathscr{S}$ is *L_p-stable* if there exist constants K_1, K_2 depending only on n, r, d and ω_Δ, such that for any $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$,

$$K_1 \|\alpha\|_{\ell_p} \leqslant \left\| \sum_{k=1}^m \alpha_k \tilde{s}_k \right\|_{L_p(\Omega)} \leqslant K_2 \|\alpha\|_{\ell_p}.$$

To establish stability of a local basis it seems most convenient to use the following general lemma; see also [23].

LEMMA 6.2. *Let $\{s_1, ..., s_m\}$ be a γ-local basis for $\mathscr{S}$, and let $\{\lambda_1, ..., \lambda_m\} \subset \mathscr{S}^*$ be its dual basis. Suppose that*

$$\|s_k\|_{L_\infty(\Omega)} \leqslant C_1, \qquad k = 1, ..., m, \tag{6.1}$$

and

$$|\lambda_k s| \leqslant C_2 \|s\|_{L_\infty(\text{star}^\gamma(v_k))}, \qquad \text{all} \quad s \in \mathscr{S}, \qquad k = 1, ..., m, \tag{6.2}$$

where supp $s_k \subset \text{star}^\gamma(v_k)$ *as in Definition* 5.1. *Then for any* $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$,

$$K_1 C_2^{-1} \|\alpha\|_{\ell_p} \leqslant \left\| \sum_{k=1}^m \alpha_k \frac{s_k}{|\text{supp } s_k|^{1/p}} \right\|_{L_p(\Omega)} \leqslant K_2 C_1 \|\alpha\|_{\ell_p}, \qquad 1 \leqslant p \leqslant \infty, \tag{6.3}$$

where K_1, K_2 are some constants depending only on n, r, d, γ and ω_Δ.

Proof. Let $s = \sum_{k=1}^m \alpha_k (s_k / |\text{supp } s_k|^{1/p})$. We first prove the upper bound in (6.3). Given an n-simplex $T \in \Delta$, we have by (6.1)

$$\|s|_T\|_{L_p(T)} \leqslant C_1 (\#\Sigma_T)^{1-1/p} \begin{cases} \left(\sum_{k \in \Sigma_T} |\alpha_k|^p \right)^{1/p}, & \text{if} \quad 1 \leqslant p < \infty, \\ \max_{k \in \Sigma_T} |\alpha_k|, & \text{if} \quad p = \infty, \end{cases}$$

where

$$\Sigma_T := \{k \colon T \subset \text{supp } s_k\}. \tag{6.4}$$

As in the bivariate case (see Lemmas 3.1 and 3.2 in [23]), it is not difficult to show that

$$\#\{T \in \Delta : T \subset \text{star}^\gamma(v_k)\} \leqslant \tilde{K}_1 \tag{6.5}$$

and

$$\max\left\{\frac{|\mathrm{star}^{\gamma}(v_k)|}{|T|} : T \subset \mathrm{star}^{\gamma}(v_k)\right\} \leqslant \tilde{K}_2, \tag{6.6}$$

where $\tilde{K}_1$, $\tilde{K}_2$ are some constants depending only on n, γ and ω_Δ. Hence, for $1 \leqslant p < \infty$ we have

$$\|s\|^p_{L_p(\Omega)} = \sum_{T \in \Delta} \|s|_T\|^p_{L_p(T)} \leqslant \tilde{K}_1 C_1^p (\#\Sigma_T)^{p-1} \|\alpha\|^p_{\ell_p},$$

which shows that the upper bound will be established for all $1 \leqslant p \leqslant \infty$ if we prove that $\#\Sigma_T$ is bounded by a constant depending only on n, r, d, γ and ω_Δ. To this end we note that since the basis $\{s_1, ..., s_m\}$ is γ-local, $\mathrm{supp}\, s_k \subset \mathrm{star}^{2\gamma}(v)$, for all $k \in \Sigma_T$, where v is any vertex of T. Therefore, the set $\{s_k : k \in \Sigma_T\}$ is linearly independent on $\mathrm{star}^{2\gamma}(v)$, and its cardinality $\#\Sigma_T$ does not exceed the dimension of the space of all piecewise polynomials of degree d on $\mathrm{star}^{2\gamma}(v)$, i.e., $\#\Sigma_T \leqslant N\binom{n+d}{n}$, where N is the number of n-simplices of Δ lying in $\mathrm{star}^{2\gamma}(v)$. By (6.5), N is bounded by a constant depending only on n, γ and ω_Δ, and the assertion follows.

To establish the lower bound in (6.3), we obtain by (6.2),

$$|\alpha_k| = |\mathrm{supp}\, s_k|^{1/p}\, |\lambda_k s| \leqslant C_2\, |\mathrm{supp}\, s_k|^{1/p}\, \|s\|_{L_\infty(\mathrm{star}^{\gamma}(v_k))}, \qquad k = 1, ..., m.$$

Since $\|s\|_{L_\infty(\mathrm{star}^{\gamma}(v_k))} \leqslant \|s\|_{L_\infty(\Omega)}$, this completes the proof in the case $p = \infty$. Suppose $1 \leqslant p < \infty$. By a Nikolskii-type inequality, see e.g. [27, p. 56], for some n-simplex $T_k \subset \mathrm{star}^{\gamma}(v_k)$,

$$\|s\|_{L_\infty(\mathrm{star}^{\gamma}(v_k))} = \|s|_{T_k}\|_{L_\infty(T_k)} \leqslant \tilde{K}_3\, |T_k|^{-1/p}\, \|s|_{T_k}\|_{L_p(T_k)},$$

where $\tilde{K}_3$ is a constant depending only on n and d. Since $\mathrm{supp}\, s_k \subset \mathrm{star}^{\gamma}(v_k)$, we have by (6.6),

$$\frac{|\mathrm{supp}\, s_k|}{|T_k|} \leqslant \tilde{K}_2.$$

Therefore,

$$\sum_{k=1}^{m} |\alpha_k|^p \leqslant \tilde{K}_2 (\tilde{K}_3 C_2)^p \sum_{k=1}^{m} \int_{T_k} |s|^p.$$

We now have to bound the number of appearances of a given n-simplex T_k on the right-hand side of the above inequality. If $T_{k_1} = T_{k_2}$, then $\mathrm{star}^\gamma(v_{k_1}) \cap \mathrm{star}^\gamma(v_{k_2}) \neq \emptyset$. Hence, $\mathrm{supp}\, s_{k_2} \subset \mathrm{star}^{3\gamma}(v_{k_1})$. Thus, for all k such that $T_k = T_{k_1}$,

$$\mathrm{supp}\, s_k \subset \mathrm{star}^{3\gamma}(v_{k_1}).$$

The set $\{s_k : T_k = T_{k_1}\}$ is linearly independent on $\mathrm{star}^{3\gamma}(v_{k_1})$, and it can be shown as above that its cardinality is bounded by a constant $\tilde{K}_4$ depending only on n, γ and ω_Δ. Therefore,

$$\sum_{k=1}^{m} \int_{T_k} |s|^p \leqslant \tilde{K}_4 \int_\Omega |s|^p,$$

which completes the proof. ▮

We are ready to formulate our main result about stability of the local basis constructed in Section 5. For each $\tau \in \mathscr{T}$, denote by h_τ the *diameter* of the set $\mathrm{star}(\tau)$. (This is compatible with the above notation h_T for $T \in \mathscr{T}_n = \Delta$ since $\mathrm{star}(T) = T$.)

THEOREM 6.3. *Suppose that*

(1) *every* $\Xi_{\tau, q}$, $q = 0, ..., q_\ell$, $\tau \in \mathscr{T}_\ell$, $1 \leqslant \ell \leqslant n$ (*where* $\Xi_{T,0} := \Xi_T$ *if* $T \in \mathscr{T}_n$), *is chosen to be the set of uniformly distributed points in the interior of* τ, *as defined in* (3.5); *and*

(2) *for each* $q = 0, ..., q_\ell$ *and* $\tau \in \mathscr{T}_\ell$, $0 \leqslant \ell \leqslant n$, *the vectors*

$$a^{[\tau, q, k]} = (a_j^{[\tau, q, k]})_{j \in J_{\tau, q}}, \qquad k = 1, ..., m_{\tau, q}, \tag{6.7}$$

form an orthonormal basis for $N(C_{\tau, q})$.

Let $\tilde{s}^{[\tau, q, k]}$ *be the local basis functions for* $\mathscr{S}_d^r(\Delta)$, $d \geqslant r2^n + 1$, *constructed as in Section* 5. *Then for every* $1 \leqslant p \leqslant \infty$, *the splines*

$$h_\tau^{-q} |\mathrm{star}(\tau)|^{-1/p} \tilde{s}^{[\tau, q, k]}, \qquad k = 1, ..., m_{\tau, q},$$
$$q = 0, ..., q_\ell, \qquad \tau \in \mathscr{T}_\ell, \qquad 0 \leqslant \ell \leqslant n,$$

form an L_p-*stable local basis for* $\mathscr{S}_d^r(\Delta)$.

Proof. As shown in Section 5, the splines $\tilde{s}^{[\tau, q, k]}$ are 1-local, and $\mathrm{supp}\, \tilde{s}^{[\tau, q, k]} \subset \mathrm{star}(\tau)$. By (6.6),

$$|\mathrm{supp}\, \tilde{s}^{[\tau, q, k]}| \leqslant |\mathrm{star}(\tau)| \leqslant \tilde{K}_2 \, |\mathrm{supp}\, \tilde{s}^{[\tau, q, k]}|,$$

where $\widetilde{K}_2$ depends only on n and ω_Δ. Hence, in view of Lemma 6.2, the theorem will be established once we prove that

$$\|\tilde{s}^{[\tau, q, k]}\|_{L_\infty(\Omega)} \leqslant C_1 h_\tau^q, \tag{6.8}$$

and

$$|\tilde{\lambda}^{[\tau, q, k]} s| \leqslant C_2 h_\tau^{-q} \|s\|_{L_\infty(\mathrm{star}(\tau))}, \qquad \text{all} \quad s \in \mathcal{S}_d^r(\Delta), \tag{6.9}$$

where the constants C_1, C_2 depend only on n, r, d and ω_Δ.

We first show (6.8). Since $\operatorname{supp} \tilde{s}^{[\tau, q, k]} \subset \mathrm{star}(\tau)$, we have $\|\tilde{s}^{[\tau, q, k]}\|_{L_\infty(\Omega)} = \|\tilde{s}^{[\tau, q, k]}\|_{L_\infty(\mathrm{star}(\tau))}$. Let T be an n-simplex in $\mathrm{star}(\tau)$, and let $\mathcal{H}_T$ be the Hermite interpolation operator defined in (3.4). Since $\tilde{s}^{[\tau, q, k]}|_T = \mathcal{H}_T \tilde{s}^{[\tau, q, k]}|_T$, we have by Lemma 3.3,

$$\|\tilde{s}^{[\tau, q, k]}|_T\|_{L_\infty(T)} \leqslant \widetilde{K}_5 \max_{\eta \in \mathcal{N}(T)} h_T^{q(\eta)} |\eta \tilde{s}^{[\tau, q, k]}|,$$

where $\widetilde{K}_5$ depends only on n, r and d. Now, by (5.7), $\eta \tilde{s}^{[\tau, q, k]} = 0$ for all $\eta \in \mathcal{N}(T) \backslash \mathcal{N}_{\tau, q}$, and

$$\eta_j^{[\tau, q]} \tilde{s}^{[\tau, q, k]} = a_j^{[\tau, q, k]}, \qquad j \in J_{\tau, q}.$$

Since the vectors $a^{[\tau, q, k]}$, $k = 1, ..., m_{\tau, q}$, are orthonormal, we have $|a_j^{[\tau, q, k]}| \leqslant 1$. Taking into account that $q(\eta) = q$ for all $\eta \in \mathcal{N}_{\tau, q}$, we arrive at the estimate

$$\|\tilde{s}^{[\tau, q, k]}|_T\|_{L_\infty(T)} \leqslant \widetilde{K}_5 h_T^q \leqslant \widetilde{K}_5 h_\tau^q,$$

and (6.8) is proved.

By our hypotheses, the columns of the matrix

$$A_{\tau, q} = [a_j^{[\tau, q, k]}]_{j \in J_{\tau, q}, k = 1, ..., m_{\tau, q}} \tag{6.10}$$

are orthonormal. Hence, $A_{\tau, q}^T$ is a left inverse of $A_{\tau, q}$. By Lemma 2.5 and the proof of Theorem 5.3, it follows that the dual functional $\tilde{\lambda}^{[\tau, q, k]}$ can be computed as

$$\tilde{\lambda}^{[\tau, q, k]} = \sum_{j \in J_{\tau, q}} a_j^{[\tau, q, k]} \eta_j^{[\tau, q]}.$$

Therefore, for any $s \in \mathcal{S}_d^r(\Delta)$,

$$|\tilde{\lambda}^{[\tau, q, k]} s| = \left| \sum_{j \in J_{\tau, q}} a_j^{[\tau, q, k]} \eta_j^{[\tau, q]} s \right| \leqslant \# J_{\tau, q} \max_{j \in J_{\tau, q}} |\eta_j^{[\tau, q]} s|.$$

Given $j \in J_{\tau,q}$, let T be an n-simplex such that $\tau \subset T$ and $\eta_j^{[\tau,q]} \in \mathcal{N}(T)$. Since $\eta_j^{[\tau,q]}$ is a nodal functional of order q, we have by Markov inequality (see, e.g. [13]),

$$|\eta_j^{[\tau,q]} s| = |\eta_j^{[\tau,q]} s|_T| \leqslant \tilde{K}_6 \rho_T^{-q} \, \|s|_T\|_{L_\infty(T)} \leqslant \tilde{K}_6 \omega_\Delta^q h_T^{-q} \, \|s\|_{L_\infty(\mathrm{star}(\tau))},$$

where $\tilde{K}_6$ is a constant depending only on n and d. Since $\# J_{\tau,q} = \# \mathcal{N}_{\tau,q}$ is bounded above by a constant depending only on n, r, d and ω_Δ, the estimate (6.9) follows, and the proof is complete. ∎

It is easy to see that Theorem 6.3 remains valid for any $\Xi_{\tau,q}$ such that the Hermite interpolation operator defined by (3.4) satisfies (3.6), and for any choice of the bases (6.7) for $N(C_{\tau,q})$ such that the *condition number* of the matrix (6.10) is bounded by a constant K depending only on n, r, d and ω_Δ; compare [6]. However, there is a good reason to prefer, at least in practice, an *orthonormal basis* for $N(C_{\tau,q})$, as explained in the following remark.

Remark 6.4. There is a numerically efficient way to compute an orthonormal basis $a^{[\tau,q,k]} = (a_j^{[\tau,q,k]})_{j \in J_{\tau,q}}$, $k = 1, ..., m_{\tau,q}$, for each $N(C_{\tau,q})$, as required in the above theorem. Namely, construct by an appropriate algorithm a *singular value decomposition* $C_{\tau,q} = Q_L X Q_R^T$ of the matrix $C_{\tau,q}$, where Q_L, Q_R are orthogonal matrices, and $X = [D\ O]$, $D = \mathrm{diag}(\sigma_1, ..., \sigma_p)$, with $\sigma_1 \geqslant \cdots \geqslant \sigma_p \geqslant 0$ being the singular values of $C_{\tau,q}$, see e.g. [29]. Obviously, $m_{\tau,q}$ is equal to the number of zero columns in X (including the columns corresponding to zero singular values). Hence, the columns of the matrix $[O\ I_{m_{\tau,q}}]^T$ constitute an orthonormal basis for $N(X)$. Since $C_{\tau,q} Q_R = Q_L X$, the columns of $A_{\tau,q} = Q_R [O\ I_{m_{\tau,q}}]^T$ form the desired orthonormal basis for $N(C_{\tau,q})$. Thus, the matrix $A_{\tau,q}$ consists of the last $m_{\tau,q}$ columns of Q_R.

7. SUPERSPLINE SPACES

In this section we construct stable local bases for the superspline subspaces of $\mathcal{S}_d^r(\Delta)$.

DEFINITION 7.1. Let $\rho = (\rho_\tau)_{\tau \in \mathcal{T} \setminus (\mathcal{T}_{n-1} \cup \mathcal{T}_n)}$ be a sequence of integers satisfying

$$r \leqslant \rho_\tau \leqslant 2^{n-\ell-1}, \qquad \tau \in \mathcal{T}_\ell, \quad 0 \leqslant \ell \leqslant n-2. \tag{7.1}$$

The linear space of splines

$$\mathscr{S}_d^{r,\rho}(\Delta) := \{s \in \mathscr{S}_d^r(\Delta) : s \text{ is } \rho_\tau\text{-times differentiable across } \tau, \\ \text{for all } \tau \in \mathscr{T} \backslash (\mathscr{T}_{n-1} \cup \mathscr{T}_n)\} \tag{7.2}$$

is called a *superspline space.*

In the limiting case $\rho_\tau = 2^{n-\ell-1}$, $\tau \in \mathscr{T} \backslash (\mathscr{T}_{n-1} \cup \mathscr{T}_n)$, the superspline spaces were introduced and studied in [8–11], see also [3, 4]. In particular, local bases for $\mathscr{S}_d^{r,\rho}(\Delta)$, where $\rho_\tau = 2^{n-\ell-1}$, were constructed in [11] and [4]. For general ρ_τ, but only in the bivariate case $n=2$, the superspline spaces were explored in [22, 28] and, more recently, in [18, 19].

As we will see, our method of construction of a stable local basis can be applied to the spaces (7.2). We first have to extend the system $\mathscr{R}$ of smoothness conditions defined in (4.4)–(4.8) to a larger system $\hat{\mathscr{R}}$, by allowing a larger range of r' in (4.4) and (4.5). Namely, we include in the extended systems $\hat{\mathscr{R}}_{v,q}$ and $\hat{\mathscr{R}}_{\tau,q,\xi}$ all conditions (4.4) and (4.5), respectively, where $0 \leqslant r' \leqslant \min\{\rho_\tau, q\}$. The systems $\mathscr{R}_{F,q,\xi}$ are not enlarged, i.e., we set $\hat{\mathscr{R}}_{F,q,\xi} = \mathscr{R}_{F,q,\xi}$.

By the method of proof of Theorem 4.4 it is not difficult to establish the following analogue of it.

THEOREM 7.2. *The set $\hat{\mathscr{R}}$ is a complete system of linear relations for $\mathscr{N}$ over $\mathscr{S}_d^{r,\rho}(\Delta)$.*

It is easy to see that the matrix $\hat{C}$ of the system $\hat{\mathscr{R}}$ possesses a block diagonal structure similar to the structure of the matrix C considered in Section 5. Therefore, all results about the dimension and the local bases carry over to the superspline spaces. Thus, we have

$$\begin{aligned}\dim \mathscr{S}_d^{r,\rho}(\Delta) &= \#\mathscr{N} - \sum_{\tau \in \mathscr{T} \backslash \mathscr{T}_n} \operatorname{rank} \hat{C}_\tau \\ &= \#\mathscr{N} - \sum_{v \in \mathscr{T}_0} \sum_{q=0}^{r2^{n-1}} \operatorname{rank} \hat{C}_{v,q} \\ &\quad - \sum_{\ell=1}^{n-1} \sum_{\tau \in \mathscr{T}_\ell} \sum_{q=0}^{r2^{n-\ell-1}} \sum_{\xi \in \Xi_{\tau,q}} \operatorname{rank} \hat{C}_{\tau,q,\xi},\end{aligned}$$

where $\hat{C}_\tau$, $\hat{C}_{v,q}$ and $\hat{C}_{\tau,q,\xi}$ are the appropriate blocks of $\hat{C}$. Define the splines

$$\hat{s}^{[\tau,q,k]}, \qquad k=1,...,\hat{m}_{\tau,q}, \qquad q=0,...,q_\ell, \qquad \tau \in \mathscr{T}_\ell, \qquad 0 \leqslant \ell \leqslant n, \tag{7.4}$$

by the condition

$$\begin{aligned} \eta_j^{[\tau, q]} \hat{s}^{[\tau, q, k]} &= \hat{a}_j^{[\tau, q, k]}, && j \in J_{\tau, q}, \\ \eta \hat{s}^{[\tau, q, k]} &= 0, && \text{all} \quad \eta \in \mathcal{N} \setminus \mathcal{N}_{\tau, q}, \end{aligned} \tag{7.5}$$

where

$$\hat{a}^{[\tau, q, k]} = (\hat{a}_j^{[\tau, q, k]})_{j \in J_{\tau, q}}, \qquad k = 1, ..., \hat{m}_{\tau, q}, \tag{7.6}$$

is a basis for $N(\hat{C}_{\tau, q})$.

THEOREM 7.3. *The splines* (7.4) *form a local basis for* $\mathcal{S}_d^{r, \rho}(\Delta)$, *where* ρ *satisfies* (7.1), *and* $d \geqslant r2^n + 1$. *Moreover*,

$$\operatorname{supp} \hat{s}^{[\tau, q, k]} \subset \operatorname{star}(\tau), \tag{7.7}$$

and the dual basis (5.8) *satisfies*

$$\hat{\lambda}^{[\tau, q, k]} s = 0 \qquad \textit{for all} \quad s \in \mathcal{S}_d^r(\Delta) \qquad \textit{such that} \quad s|_{\operatorname{star}(\tau)} = 0. \tag{7.8}$$

Since (7.4) is a local basis for $\mathcal{S}_d^r(\Delta)$, Lemma 6.2 can be applied, and the same argument as in the proof of Theorem 6.3 shows that the following result holds.

THEOREM 7.4. *Suppose that*

(1) *every* $\Xi_{\tau, q}$, $q = 0, ..., q_\ell$, $\tau \in \mathcal{T}_\ell$, $1 \leqslant \ell \leqslant n$ (*where* $\Xi_{T, 0} := \Xi_T$ *if* $T \in \mathcal{T}_n$), *is chosen to be the set of uniformly distributed points in the interior of* τ, *as defined in* (3.5), *and*

(2) *for each* $q = 0, ..., q_\ell$ *and* $\tau \in \mathcal{T}_\ell$, $0 \leqslant \ell \leqslant n$, *vectors* $\hat{a}^{[\tau, q, k]} = (\hat{a}_j^{[\tau, q, k]})_{j \in J_{\tau, q}}$, $k = 1, ..., m_{\tau, q}$, *form an orthonormal basis for* $N(\hat{C}_{\tau, q})$.

Let $\hat{s}^{[\tau, q, k]}$ *be the local basis functions* (7.4) *for* $\mathcal{S}_d^{r, \rho}(\Delta)$, *where* ρ *satisfies* (7.1), *and* $d \geqslant r2^n + 1$. *Then for every* $1 \leqslant p \leqslant \infty$, *the splines*

$$\begin{gathered} h_\tau^{-q} \, |\operatorname{star}(\tau)|^{-1/p} \, \hat{s}^{[\tau, q, k]}, \qquad k = 1, ..., m_{\tau, q}, \\ q = 0, ..., q_\ell, \qquad \tau \in \mathcal{T}_\ell, \qquad 0 \leqslant \ell \leqslant n, \end{gathered}$$

form an L_p-*stable local basis for* $\mathcal{S}_d^{r, \rho}(\Delta)$.

ACKNOWLEDGMENTS

The author is grateful to the editor of this paper and to a referee for helpful suggestions for improving the manuscript and for pointing out a number of misprints in its original version.

REFERENCES

1. P. Alfeld, B. Piper, and L. L. Schumaker, Minimally supported bases for spaces of bivariate piecewise polynomials of smoothness r and degree $d \geqslant 4r+1$, *Comput. Aided Geom. Design* **4** (1987), 105–123.
2. P. Alfeld, L. L. Schumaker, and M. Sirvent, On dimension and existence of local bases for multivariate spline spaces, *J. Approx. Theory* **70** (1992), 243–264.
3. P. Alfeld and M. Sirvent, A recursion formula for the dimension of super spline spaces of smoothness r and degree $d > r2^k$, *in* "Multivariate Approximation Theory IV, ISNM 90" (C. Chui, W. Schempp, and K. Zeller, Eds.), pp. 1–8, Birkhäuser, Basel, 1989.
4. P. Alfeld and M. Sirvent, The structure of multivariate superspline spaces of high degree, *Math. Comp.* **57**(195) (1991), 299–308.
5. C. de Boor, *B*-form basics, *in* "Geometric Modeling: Algorithms and New Trends" (G. E. Farin, Ed.), pp. 131–148, SIAM, Philadelphia, 1987.
6. C. de Boor, A local basis for certain smooth bivariate pp spaces, *in* "Multivariate Approximation Theory IV, ISNM 90" (C. Chui, W. Schempp, and K. Zeller, Eds.), pp. 25–30, Birkhäuser, Basel, 1989.
7. C. K. Chui, D. Hong, and R.-Q. Jia, Stability of optimal order approximation by bivariate splines over arbitrary triangulations, *Trans. Amer. Math. Soc.* **347** (1995), 3301–3318.
8. C. K. Chui and M.-J. Lai, On bivariate vertex splines, *in* "Multivariate Approximation Theory III, ISNM 75" (W. Schempp and K. Zeller, Eds.), pp. 84–115, Birkhäuser, Basel, 1985.
9. C. K. Chui and M.-J. Lai, On bivariate super vertex splines, *Constr. Approx.* **6** (1990), 399–419.
10. C. K. Chui and M.-J. Lai, On multivariate vertex splines and applications, *in* "Topics in Multivariate Approximation" (C. K. Chui, L. L. Schumaker, and F. Utreras, Eds.), pp. 19–36, Academic Press, New York, 1987.
11. C. K. Chui and M.-J. Lai, Multivariate vertex splines and finite elements, *J. Approx. Theory* **60** (1990), 245–343.
12. P. G. Ciarlet, "The Finite Element Method for Elliptic Problems," North-Holland, The Netherlands, 1978.
13. P. G. Ciarlet and P. A. Raviart, General Lagrange and Hermite interpolation in $\mathbb{R}^N$ with applications to finite element methods, *Arch. Rational Mech. Anal.* **46** (1972), 177–199.
14. W. Dahmen, P. Oswald, and X.-Q. Shi, C^1-hierarchical bases, *J. Comput. Appl. Math.* **51** (1994), 37–56.
15. O. Davydov, Locally linearly independent basis for C^1 bivariate splines, *in* "Mathematical Methods for Curves and Surfaces II" (M. Dæhlen, T. Lyche, and L. Schumaker, Eds.), pp. 71–78, Vanderbilt University Press, Nashville/London, 1998.
16. O. Davydov, G. Nürnberger, and F. Zeilfelder, Bivariate spline interpolation with optimal approximation order, *Constr. Approx.* **17** (2001), 181–208.
17. O. Davydov and L. L. Schumaker, Stable local nodal bases for C^1 bivariate polynomial splines, *in* "Curve and Surface *Fitting: Saint-Malo 1999" (A. Cohen, C. Rabut, and L. L. Schumaker, Eds.), pp. 171–180, Vanderbilt University Press, Nashville, TN, 2000.
18. O. Davydov and L. L. Schumaker, Locally linearly independent bases for bivariate polynomial splines, *Adv. Comput. Math.* **13** (2000), 355–373.

19. O. Davydov and L. L. Schumaker, On stable local bases for bivariate polynomial spline spaces, *Constr. Approx.*, to appear.
20. G. Farin, Triangular Bernstein–Bézier patches, *Comput. Aided Geom. Design* **3** (1986), 83–127.
21. D. Hong, Spaces of bivariate spline functions over triangulation, *Approx. Theory Appl.* **7** (1991), 56–75.
22. A. Ibrahim and L. L. Schumaker, Super spline spaces of smoothness r and degree $d \geqslant 3r+2$, *Constr. Approx.* **7** (1991), 401–423.
23. M.-J. Lai and L. L. Schumaker, On the approximation power of bivariate splines, *Adv. Comput. Math.* **9** (1998), 251–279.
24. A. Le Méhauté, Unisolvent interpolation in $\mathbb{R}^n$ and the simplicial polynomial finite element method, *in* "Topics in Multivariate Approximation" (C. K. Chui, L. Schumaker, and F. Utreras, Eds.), pp. 141–151, Academic Press, New York, 1987.
25. A. Le Méhauté, Nested sequences of triangular finite element spaces, *in* "Multivariate Approximation: Recent Trends and Results" (W. Haussman, K. Jetter, and M. Reimer, Eds.), pp. 133–145, Akademie-Verlag, Berlin, 1997.
26. J. Morgan and R. Scott, A nodal basis for C^1 piecewise polynomials of degree $n \geqslant 5$, *Math. Comp.* **29**(131) (1975), 736–740.
27. P. Oswald, "Multilevel Finite Element Approximation," Teubner, Stuttgart, 1994.
28. L. L. Schumaker, On super splines and finite elements, *SIAM J. Numer. Anal.* **26** (1989), 997–1005.
29. G. W. Stewart, "Matrix Algorithms, Volume I: Basic Decompositions," SIAM, Philadelphia, 1998.
30. A. Ženišek, Polynomial approximation on tetrahedrons in the finite element method, *J. Approx. Theory* **7** (1973), 334–351.